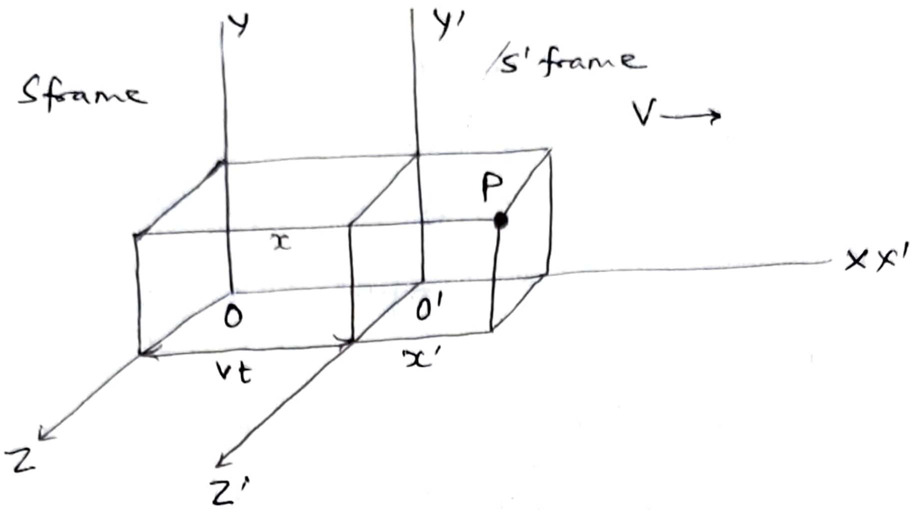
Relativity

BSc I(H)

Lorentz transformation Equation



The relation between the observations of position and time made by two observers in two different inertial frames are known as Lorentz transformation.

Let S and S' are two inertial frames of reference, S' is moving with velocity v relative to S. Consider that the X-axes of two systems coincides permanantly and velocity is parallel to X-axis. The event P is a light signal and is produced when both t and t' are zero and when origins of the two frame coincide. The event P is determined by coordinates (x, y, z, t) and (x', y', z', t') by observers O and O' respectively. The space is isotropic.

We assume that space and time are homogeneous which requires that transformation equation must be linear. We have

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$$

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

We use the postulates of relativity

1. No preferred inertial system exists, the law of physics being the same in all inertial systems.

2. The speed of light in free space has the same value C in all inertial system.

Since S' is moving along common X-axis, therefore, y' and z' cannot depend on t. Also, axes of S and S' frames are parallel. x' cannot depend on y and z and for the same reason y' and z' are independent of x, z and x, y respectively.

i.e. Coefficients $a_{21}, a_{24}, a_{31}, a_{34}, a_{23}$ and a_{32} are zero, therefore from symmetry, our equation will be

$$y' = a_{22} y$$

$$\text{and } z' = a_{33} z \qquad \text{——— (2)}$$

For a_{22}, we suppose a rod lying along y-axis, measured by S to be of unit length. According to the S' observer the rods length will be a_{22} (i.e. $y' = a_{22} \times 1$). Now suppose the same rod is brought to rest along the y' axis in S' frame. The primed observer must measure the same length (unity) for this rod, when it is rest in his frame. In this case, the S observer would measure the rods length to be $\frac{1}{a_{22}}$ $\left[\text{i.e. } y = \left(\frac{1}{a_{22}}\right) y' = \frac{1}{a_{22}} \times 1\right]$

Now, because of reciprocal nature of these length measurements, the first postulates requires that these measurements be identical, otherwise the frame would not be equivalent physically. Hence, we must have $\frac{1}{a_{22}} = a_{22}$ or $a_{22} = 1$.

Similarly, we determine $a_{33}=1$.

Therefore, our transformation equation becomes

$$y'=y \text{ and } z'=z \quad \text{——— (3)}$$

For t' equation

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

we assume that t' does not depend on y and z otherwise, Clock placed symmetrically in y-z plane about the x-axis would appear to disagree as observed from S', which would contradict the isotropy of space. Hence, $a_{42}=a_{43}=0$

For x' equation

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

we know that a point having $x'=0$ appears to move in the direction of positive x-axis with Speed V, so that the statement $x'=0$ must be identical to the statement $x=Vt$. Therefore, we expect $x'=a_{11}(x-Vt)$ to be the correct transformation equation (i.e, $x=vt$ always gives $x'=0$ in this equation.)

Hence,

$$x' = a_{11}x - a_{11}vt = a_{11}x + a_{14}t$$

This gives us $a_{14}=-Va_{11}$ and four equation reduced to

$$x' = a_{11}(x-vt)$$
$$y'=y$$
$$z'=z \quad \text{——— (4)}$$
$$t' = a_{41}x + a_{44}t$$

Now, Let us assume that at the time $t=0$, a spherical e.m. wave leaves the origin of S, which coincides with the origin of S' at that moment. The wave propagates with a speed c in all direction in each inertial frames. Its progress is described by equation of sphere whose radius expands with time at a rate c in terms of either primed or unprimed set of Coordinates ie,

$$x^2+y^2+z^2 = c^2t^2 \quad \text{--- (5)}$$
$$x'^2+y'^2+z'^2 = c^2t'^2 \quad \text{--- (6)}$$

Substituting eqn. (4) in eqn. (6) we get

$$a_{11}^2(x-vt)^2+y^2+z^2 = c^2(a_{41}x+a_{44}t)^2$$

$$\Rightarrow a_{11}^2(x^2-2xVt+v^2t^2)+y^2+z^2 = c^2(a_{41}^2x^2+a_{44}^2t^2+2a_{41}x\,a_{44}t)$$

$$\Rightarrow a_{11}^2x^2 - a_{11}^2\,2xvt + a_{11}^2V^2t^2+y^2+z^2 = c^2a_{41}^2x^2 + c^2a_{44}^2t^2 + 2a_{41}x\,a_{44}t\,c^2$$

$$= (a_{11}^2 - c^2a_{41}^2)x^2+y^2+z^2-2(Va_{11}^2+c^2a_{41}a_{44})xt = (c^2a_{44}^2-v^2a_{11}^2)t^2$$

Comparing this equation with eqn. (5), we get

$$a_{11}^2 - c^2a_{41}^2 = 1 \quad \text{--- (7a)}$$

$$c^2a_{44}^2 - v^2a_{11}^2 = c^2 \quad \text{--- (7b)}$$

$$Va_{11}^2 + c^2a_{41}a_{44} = 0 \quad \text{--- (7c)}$$

Solving these equation

From (7c) eliminate a_{11} and put in (7a), we get

$$\left(-\frac{c^2}{v}\right)a_{41}a_{44} - c^2a_{41}^2 = 1$$

$$\Rightarrow -c^2a_{41}a_{44} - c^2Va_{41}^2 = V$$

$$\Rightarrow c^2a_{41}(a_{44}+Va_{41}) = -V \quad \text{--- (8a)}$$

Now putting values in (7b), we get

$$c^2a_{44}^2 - v^2\left(\frac{-c^2}{v}a_{41}a_{44}\right) = c^2$$

$$vc^2 a_{44}^2 + v^2c^2 a_{41} a_{44} = vc^2$$

$$a_{44}^2 + v a_{41} a_{44} = 1$$

$$\Rightarrow a_{44}(a_{44} + v a_{41}) = 1 \quad \text{—— (8b)}$$

From (8a) and (8b), we get

$$c^2 a_{41} = -v a_{44} \quad \text{—— (8c)}$$

Substituting eqn. (8c) in eqn. (8b), we get

$$a_{44}\left(a_{44} + \frac{v(-v)a_{44}}{c^2}\right) = 1$$

$$\Rightarrow a_{44}\left(a_{44} - \frac{v^2}{c^2} a_{44}\right) = 1$$

$$a_{44}^2\left(1 - \frac{v^2}{c^2}\right) = 1$$

$$a_{44} = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{—— (9)}$$

Also from eqn. (7c)

$$v a_{11}^2 = -c^2 a_{41} a_{44}$$

$$\Rightarrow v a_{11}^2 = -(-v a_{11}) a_{44}$$

$$\Rightarrow a_{11}^2 = a_{44}^2$$

$$\Rightarrow a_{11} = a_{44} \Rightarrow a_{11} = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{— (10)}$$

Similarly we get

$$a_{41} = \frac{-v/c^2}{\sqrt{1 - v^2/c^2}} \quad \text{—— (11)}$$

Substituting the values from eqn. (9), (10) & (11), we get

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z \qquad \text{—— (12)}$$

$$t' = \frac{t - \left(\frac{v}{c^2}\right)x}{\sqrt{1 - v^2/c^2}}$$

This is Lorentz transformation equation
The inverse Lorentz transformation is given by

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$y = y'$$

$$z = z' \qquad \text{—— (13)}$$

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - v^2/c^2}}$$

Since $v \ll c$ i.e., for $\frac{v}{c} \ll 1$, the Lorentz transformation eqn. should ~~eq~~ reduce to correct Gallilean transformation eqn. when $\frac{v}{c} \ll 1$ eqn. (11) becomes

$$x' = x - vt$$

$$y' = y$$

$$z' = z \qquad \text{—— (14)}$$

$$t' = t$$

Which are the classical transformation equation.

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Dr. Shailaj Kumar Shrivastava
Principal